Algebra II **Topic:** Matrices

By: Marley, Grayson, Kaleb, Josh R. and Nick T.

Definitions:

[1] Matrices are used to organize data in columns and rows with brackets. The values entered into the brackets are known as elements or entries.

[2] A matrix is a group of mathematical numbers, symbols or glyphs with some value that is grouped into a pair and placed within brackets or parentheses. They are used to solve mathematical problems.

[3] A matrix is a rectangular arrangement of numbers or symbols used in math to represent and work with linear transformations. Matrices are used in graphic programming because a single matrix can contain all of the values to translate, scale, and rotate a shape

[4] A matrix is a mathematical rectangle for organizing and figuring out specific dimensions and specific numbers.

[5] A matrix (or matrices) is a two-dimensional way of representing data in an arrangement (array) of numbers -called elements- in rows and columns (contained by brackets on either side). Matrices can be added, subtracted, multiplied, and reduced. A matrix also has an inverse if it is a square matrix (2 by 2 for example).

Professions Utilizing Matrices:

[1] In agriculture, crop rotation has really revolutionized the industry. Due to the fact that farmers don't want one field to become full of overused soil, they have to find a way to rotate the crops to allow fields to have a rest year. They use matrices to organize the crops and rotate them in a fashion that allows for each field to have a year that is crop-free.

[2] Physicists use matrices to simplify and contain what would otherwise be sprawling equations and to

figure out how well certain designs work. For example, physicists use matrices to set up optical systems for telescopes used in observatories and on satellites so that the earth and the universe can be observed.

[3] The study of human populations is known as demography. Demographers use statistics to monitor the size of a population over time as well as to monitor separate groups within the population as defined by age, religion or ethnicity. To organize this information, demographers use matrices. Matrices can also be used by ecologists to observe populations of animals and plants as well.

[4] Computer programmers use matrices to determine rotation degrees and angles when working with 2-D images.

[5] Matrices can also be applied to the field of cryptography (the art of writing or solving codes) to both encrypt and decrypt hidden messages.

Skills:

People in many professions use math and matrices every day, whether or not they realize it. Matrices are used to organize information and data in brackets. Matrices can be complicated, but they really come down to basic elemental arithmetic skills. To organize the data, you must be able to read tables and understand the difference between rows and columns. Additionally, you should be able to perform addition, subtraction, and multiplication with integers, fractions, and variables. This includes multiplying by (real numbers that relate to vectors and can be multiplied by a number(s) to produce another vector). To perform any of the operations above accurately with matrices, it is imperative to be able to keep track of place value and the locations of numbers.

For multiplication, the number of columns in the first matrix must be the same as the number of rows in the second matrix in order for them to be multiplied together. Thus, a 4x5 matrix could be multiplied by a 5x2 matrix. The number of rows in the first matrix is the number of rows in the final and the number of columns in the second matrix is the number of columns in the final. Another way to multiply matrices is through scalar multiplication. This approach is where a number (called an element) is placed before the matrix outside of the brackets and every element gets multiplied by that number. This it is why a basic knowledge of the distributive property will aid in this process. The distributive property is, in essence, separating or combining a value(s) into parts, making them easier to work with.

To use matrices, it is also important to understand what variables are and how substitution works. Variables are sometime used in a matrix to mark an unknown value. If that value was to become known, you would have to be able to substitute the number for the variable.

Like many mathematical concepts, matrices can be difficult and many more rules come into play with more advanced matrices. Being able to think logically about situations and problems would be of great utilization when working with matrices. Remember to take your time and practice, practice, practice!

Examples:

[1] Crop Rotation Management

With modern technology, crop rotation has become a very important way to monitor fields. Farmers use matrix multiplication to determine which fields to rotate.

To find a rotation matrix, start with a 4x4 matrix. Then fill the matrix with 0's and 1's. In each column and row, there should be 3 zeros and one 1. These values can be rearranged in any pattern as long as there is only one 1 in each column and row.

Year 1 Crop Rotation

A	B	C	X
B	C	X	A
X	A	B	C

Year 2 Crop Rotation

B	A	X	C
C	B	A	X
A	X	C	B

(“X” means no crop so the field can rest for a year)

In the above example, the crops are currently lined up so that no crop is in the same field as the previous year and no two adjacent fields have the same crop. Farmers come up with this result by multiplying the year one rotation by a rotation matrix.

Crop Arrangement Rotation Matrix New Crop Arrangement

$$\begin{matrix} A & B & C & X \\ B & C & X & A \\ X & A & B & C \end{matrix} \times \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix} = \begin{matrix} B & A & X & C \\ C & B & A & X \\ A & X & C & B \end{matrix}$$

With a rotation matrix, the fields can be rotated in a simple way. For a new year, you take the rotation matrix and flip it so you would have a matrix that read:

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

Then, take the year 2 crop arrangement and multiply by the new rotation matrix to get a third year crop arrangement (“Crop Rotation”).

[2] Physics’s Using Optical Systems and Ray Matrices

Ray matrices are used to determine the image, input and output planes on optical systems. This example below shows how ray matrices are used to determine output and maximum power for optical systems and displays.

To complete an equation such as this requires some degree of work, but ultimately begins with an optical ray (r) on plane *P*, a distance *z* along the optical axis by a two component vector:

$$\mathbf{r} = \begin{bmatrix} r \\ \theta \end{bmatrix}$$

We then describe the propagation of this ray through a 2x2 matrix:

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The height and angle of this new ray is given through this next equation:

$$\mathbf{r}_1 = \mathbf{M}\mathbf{r}_0 \quad \Rightarrow \quad \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix}$$

We then define a Matrix Operator **M** for each optical *component*[2]. We can trace rays through the system by a series of matrix multiples. If we have *N* components with propagation, Matrices for $\mathbf{M}_i$ for **i=1,...,N**, then for an initial ray of r0 the final ray of rN is given by a series of *pre-multiplies*, giving:

$$\mathbf{r}_N = \mathbf{M}_N \mathbf{M}_{N-1} \ldots \mathbf{M}_2 \mathbf{M}_1 \mathbf{r}_0$$

Which we can write as:

$$\mathbf{r}_N = \mathbf{M}_s \mathbf{r}_0 \quad \text{where} \quad \mathbf{M}_s = \mathbf{M}_N \mathbf{M}_{N-1} \ldots \mathbf{M}_2 \mathbf{M}_1$$

Which allows us to form a single *system matrix* and subsequently form a combination of optical components by a series of 2x2 matrix multiples. This reduces to geometrical optical properties of a potentially very complex optical system to a simple 2x2 matrix (“Optics”).

[3] Population Projection

A projection matrix is a square matrix with any value of rows and columns. The matrix that is used is called a Leslie matrix. The Leslie matrix is an age-structured model of population growth that is very popular in population ecology. (“Leslie”) A vector matrix is used to determine the prediction for the next year's population. The vector matrix can contain one column and any amount of rows.

Leslie Matrix:

F_0	F_1	F_2
S_0	0	0
0	S_1	0

Fx = age-specific fecundity

Sx = age-specific survival rate (Montana, 1).

Values can be plugged into this matrix for any population. For a population of female frogs, we can use a vector matrix to determine the amount of frogs for any year for each stage of the frog. (pre-juvenile (PJUV), juvenile (JUV), and adult (AD)).

Year One Matrix

	PJUV	JUV	AD
PJUV	0	52	279.5
JUV	0.024	0.25	0
AD	0	0.08	0.43

Element *JUV*, (0.024) means that, on average, 2.4% of the pre-juvenile female frogs in the population survive to become juveniles the next year. Each element on the top row of the matrix (that is, elements 0, 52, 279.5 in our example matrix) represents the reproductive contribution of each stage to the next time step. Only adult females are capable of laying eggs. Thus, pre-juvenile females, because they don't reproduce, do not contribute any individuals to the population. The proportion of juvenile females in the population that will survive to adulthood contribute, on average, 52 individuals, and adult females contribute 279.5 individuals. All the elements in the matrix except for the first row represent survival rates. We said above that 2.4% of pre-juvenile female frogs survive to become juveniles. Furthermore, 25% of juvenile females survive to remain juveniles, 8% of juvenile females survive to become adults, and 43% of adult females survive to remain adults.

When we multiply the appropriate survival rate by the mean number of eggs a female frog will produce annually (650 eggs). We can calculate the reproductive contribution of each stage to the next time step. For example, we know that 8% of juvenile females survive to become adults. Multiplying 0.08 by 650 = 52, the number of eggs juvenile females transition into adulthood will contribute to the population. We also know that 43% of adult females survive to remain adults. Multiplying 0.43 by 650 = 279.5, the number of eggs adult females will contribute to the population.

To determine the population-size vector next year [n(t+1)], multiply the matrix M of vital rates by the vector of individuals at time t, n(t): n(t+1) = M x n(t). Our initial population-size vector is:

70
20
10

Which means we have a total of 100 female frogs in our population: 70 are prejuveniles, 20 are juveniles, and 10 are adults.

To find the next years population, we multiply the "year one population size" matrix by the vector matrix. The end result is the prediction for the first year. To continue the predictions, the "year one population size" matrix is multiplied by the result of the first multiplication problem. ("Arkansas," 1-2)

[4] Programming and Calculating Shapes

Computer programmers are able to create a shape by defining its vertices in Cartesian coordinates and mapping them. When lines are drawn between these coordinates, a shape is formed. However, code must be created in order to manipulate the shapes in various ways so that they can be drawn anywhere on the screen in any desired orientation (by using rotation and transformation calculations). Various kinds of transformations can be calculated before the object is finally drawn on the screen. Conducting so many calculations is time consuming, so matrices are employed to make the process of transformations less arduous.

```
MATRIX3X3 m;

  m[0][0] = cos(radians); m[0][1] = sin(radians); m[0][2] = 0.0;

  m[1][0] = -sin(radians); m[1][1] = cos(radians); m[1][2] = 0.0;

  m[2][0] = 0.0;           m[2][1] = 0.0;           m[2][2] = 1.0;
```

A single matrix can be made up of all the values that need to be calculated at the same time. For example, the matrix above can be used to calculate the angle of rotation in radians of a shape and its coordinates (Walnum, p4).

[5] Cryptography with Matrices

A message can be encoded using an encoding matrix and the recipient can decode the message by using the inverse of the matrix, called the decoding matrix.

First a message - or string of characters - is written and a number value is assigned to each letter. A is 1, B is 2, C is 3, Z is 26, and so on. A space between words is assigned the value of 27.

After the phrase is written out using the above numbers, the values are then written into a matrix (usually with three rows). An encoding matrix, or a matrix that has a random assortment of values (with the number of rows matching the message matrix; three in this case) is multiplied by the message matrix.

The resulting matrix is then sent as the message. The receiver then uses the inverse of the encoding matrix to decrypt the message by multiplying the two together. The resulting matrix is then written out and transcribed back over to letters (e.g., a 3 would be C and 25 would be Y) (“Application to Cryptography”; "Cryptography With Matrices”; Forester).

Works Cited:

“6th Grade Skills in Math." Helping With Math. 2013. Web. 5 Mar. 2014. <http://www.helpingwithmath.com/by_grade/gr6_cc_skills.htm>.

"An Introduction To Matrices." ping.com. Web. 15 Apr. 2013. <http://www.ping.be/~ping1339/matr.htm>.

Arkansas Forest Resource Center. Arkansas Forest Resource Center, 2007. Web. 29 Apr. 2013. <http://www.afrc.uamont.edu/whited/Population%20projection%20models.pdf>.

"Application to Cryptography." UOttawa. University of Ottawa. Web. 5 Mar. 2014. <http://aix1.uottawa.ca/~jkhoury/cryptography.htm>.

"Careers in Mathematics." Vanier College. Vanier College. Web. 11 Mar. 2014. <http://fclass.vaniercollege.qc.ca/web/mathematics/about/careers.htm>.

"Crop Rotation." The Future Channel. The FUTURES Channel, 2000. Web. 11 Mar. 2014. <http://www.thefutureschannel.com/pdf/algebra/crop_rotation.pdf>.

"Cryptography With Matrices." Tutorials, Quizzes, and Help. Sophia Learning. Web. 5 Mar.

2014. <http://www.sophia.org/concepts/cryptography-with-matrices>.

“Demography.” Wikipedia, 26 Apr. 2013. Web. 29 Apr. 2013. <http://en.wikipedia.org/wiki/Demography#Science_of_population>.

Forester, Christine M., and Christine A. Rivecco. "Matrices: A Secret Weapon." National Security Agency. National Security Agency, 15 Jan. 2009. Web. 6 Mar. 2014. <http://www.nsa.gov/academia/_files/collected_learning/high_school/algebra/matrices_secret_weapon.pdf>.

Headley-Walker, Linda. "Math Matrix." Math Matrix. New York State Adult Education Resource Guide and Learning Standards. Web. 11 Mar. 2014. <http://tech.worlded.org/docs/maththing/ny1p8.htm>.

Hui, T. L. "XP Math - Jobs That Use." *XP Math - Jobs That Use*. 2001. Web. 24 Apr. 2013. <http://www.xpmath.com/careers/topicsresult.php?subjectID=4>.

“Leslie Matrix.” Wikipedia, 28 Feb. 2013. Web. 29 Apr. 2013. <http://en.wikipedia.org/wiki/Leslie_matrix>.

"Linear Algebra: Matrices." Colorado University, Web. 15 Apr. 2013. <http://www.colorado.edu/engineering/cas/courses.d/IFEM.d/IFEM.AppB.d/IFEM.AppB.pdf>.

"Linear Transformations as Matrix Vector Products | Functions and Linear Transformations |Khan Academy." *Khan Academy*. Khan Academy, Web. 27 Apr. 2013. <http://www.khanacademy.org/math/linear-algebra/matrix_transformations/linear_transformations/v/linear-transformations-as-matrix-vector-products>.

"Matrices." dictionary.com. Web. 15 Apr. 2013.

<http://dictionary.reference.com/browse/matrices>.

"Matrices." Khan Academy. Khan Academy. Web. 11 Mar. 2014. <https://www.khanacademy.org/math/algebra/algebra-matrices>.

Montana State University. Montana State University. Web. 29 Apr. 2013. <http://www.montana.edu/wwwbi/staff/creel/bio480/leslie.pdf>.

Morris, Vernon. "Matrices." Math Warehouse. Web. 6 Mar. 2014. <http://www.mathwarehouse.com/algebra/matrix/real-world-applications-matrix.php>.

Nehme, Zeina. "Matrices Used in Physics." Applications of Linear Algebra. Notre Dame University, 26 Apr. 2011. Web. 11 Mar. 2014. <http://ndu2009algebra.blogspot.com/2011/04/matrices-used-in-physics.html>.

Optics: "Topic 3: Matrix Geometric Optics." www2.ph.ed.ac.uk. Web. 19 Apr. 2013. <http://www2.ph.ed.ac.uk/~wjh/teaching/optics/documents/matrix.pdf>.

Pierce, Rod. "Matrices" Math Is Fun. Ed. Rod Pierce. 26 Jan 2014. 6 Mar 2014 <http://www.mathsisfun.com/algebra/matrix-introduction.html>.

Stahler, Wendy. "Chapter 5. Matrix Operations." N. pag. Beginning Math and Physics for Game Programmers. New Riders, 2004. Web. 11 Mar. 2014. <http://flylib.com/books/en/3.188.1.41/1/>.

Stapel, Elizabeth. "Introduction to Matrices / Matrix Size." *Introduction to Matrices / Matrix Size*. Web. 24 Apr. 2013. <https://www.purplemath.com/modules/matrices.htm>.

SparkNotes Editors. “Matrices.” SparkNotes.com. SparkNotes LLC. Web. 6 Mar. 2014. <http://www.sparknotes.com/math/algebra2/matrices/summary.html>.

Walnum, Clayton. "Programming 2D Computer Graphics." *InformIT: The Trusted Technology*

Source for IT Pros and Developers. Pearson, 4 July 2003. Web. 26 Apr. 2013. <http://www.informit.com/articles/article.aspx?p=98117&seqNum=4>.

Weisstein, Eric. "Matrix." Wolfram MathWorld. Wolfram Research. Web. 6 Mar. 2014. <http://mathworld.wolfram.com/Matrix.html>.